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Note

Neighborhood unions and factor critical graphs

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Abstract

A graph G is said to be n-factor-critical if $G-T$ has a perfect matching for each $T \subset V(G)$ with $|T|=n$. In this note we give a sufficient condition for a graph to be n-factor-critical. Let G be a k-connected graph of order p, and let n be an integer with $0 \leqslant n \leqslant k$ and $p \equiv n \pmod 2$ and α be a real number with $\frac{1}{2} \leqslant \alpha \leqslant 1$. We prove that if $|N_G(A)| > \alpha(p-2k+n-2)+k$ for every independent set A of G with $|A| = \lfloor \alpha(k-n+2) \rfloor$, then G is n-factor-critical. We also discuss the sharpness of the result and the relation with matching extension.

Keywords: n-factor-critical graphs; n-extendable graphs; Neighborhood union

In this note we only consider finite simple graphs. For a vertex x in a graph G, the neighborhood of x in G is denoted by $N_G(x)$. We also consider the neighborhood of a vertex set. For $A \subset V(G)$, we define $N_G(A)$ by $N_G(A) = \bigcup_{x \in A} N_G(x)$. The number of the components of odd order in G is denoted by $\mathrm{o}(G)$. For other graph-theoretic terminology and notation, we refer the reader to [1].

In [3,4] Favaron introduced the notion of n-factor-critical graphs. For a nonnegative integer n, a graph G is said to be *n-factor-critical* if $G-T$ has a perfect matching for every $T \subset V(G)$ with $|T|=n$. There is a close relation between n-factor-criticality and n-extendability. For a nonnegative integer n, a graph G is said to be *n-extendable* if G has a matching of order n and every matching of order n extends to a perfect matching in G. Favaron [4] remarked that a $2n$-factor-critical graph is n-extendable.

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Plummer [5] gave a sufficient condition for a graph to be n-extendable in terms of neighborhood unions. Later Favaron [4] generalized it to n-factor-criticality.

Theorem A (Plummer [5]). *Let G be a k-connected graph of order p, where p is even, and let n be an integer with $0 \leqslant n < \frac{1}{2}p$ and t be an integer with $1 \leqslant t \leqslant k-2n+2$. If $|N_G(A)| \geqslant p-k+2n-1$ for every independent set A of G with $|A| = t$, then G is n-extendable.*

Theorem B (Favaron [4]). *Let G be a k-connected graph of order p, and let n and t be integers with $0 \leqslant n \leqslant k$, $n \equiv p \pmod 2$ and $1 \leqslant t \leqslant k-n+2$. If $|N_G(A)| \geqslant p-k+n-1$ for every independent set A of G with $|A| = t$, then G is n-factor-critical.*

Since $A' \subset A$ implies $N_G(A') \subset N_G(A)$, the statement of Theorem B (resp. Theorem A) is the strongest when $t = k-n+2$ (resp. $k-2n+2$). (In fact, the assumption of Theorem B (resp. Theorem A) implies that G has no independent set of order $k-n+2$ (resp. $k-2n+2$). In [4] Favaron implicitly proved that a k-connected graph G with independence number at most $k-n+1$ is n-factor-critical.)

Plummer [5] also discussed the possibility of reducing the coefficient of p from 1 to a smaller number in the bound $p-k+2n-1$ in Theorem A. He observed that the following result is an immediate corollary of the theorem on Hamiltonian-connectedness by Faudree et al. [2].

Theorem C. *Let G be a 3-connected graph of order p, where p is even. If $|N_G(u) \cup N_G(v)| \geqslant \frac{2}{3}p$ for every pair of nonadjacent vertices u and v of G, then G is 2-factor-critical, and hence 1-extendable.*

On the other hand, Plummer claimed that for $n \geqslant 2$, there does not exist a real number c with $0 < c < 1$ that guarantees the n-extendability of G when $|N_G(u) \cup N_G(v)| \geqslant cp$ for all pairs of nonadjacent vertices u, v in G of even order p. However, if we consider n-extendability or n-factor-criticality, it is natural to introduce the effect of n into the condition. The purpose of this note is to show that by introducing a term on n, we can reduce the coefficient of p from 1 down to $\frac{1}{2}$. Although we give a more general result later, we present the following theorem so that we can compare it with Theorem C.

Theorem 1. *Let G be a k-connected graph of order p, and let n be an integer with $0 \leqslant n \leqslant k-2$ and $p \equiv n \pmod 2$. If $|N_G(u) \cup N_G(v)| \geqslant \frac{1}{2}(p+n)$ for each pair of nonadjacent vertices u, $v \in V(G)$, then G is n-factor-critical.*

Actually, we do not have to restrict ourselves to a pair of nonadjacent vertices, or an independent set of order two. We can extend Theorem 1 in the following way.

Theorem 2. *Let G be a k-connected graph of order p, and let n be an integer with $0 \leqslant n \leqslant k$ and $p \equiv n \pmod 2$. If $|N_G(A)| \geqslant \frac{1}{2}(p+n)$ for every independent set A of G with $|A| = \lfloor \frac{1}{2}(k-n+2) \rfloor$, then G is n-factor-critical.*

If we compare Theorem 2 with Theorem B, we notice that the coefficient of p in the bound given to $|N_G(A)|$ is reduced from 1 to $\frac{1}{2}$. But we also note that the order of the independent set is also reduced by half. Roughly speaking, it is a price which we have to pay to obtain the better bound. On the other hand, we can increase the order of the independent sets by increasing the coefficient of p. Now we present the main theorem of this note, which immediately implies both Theorem B and Theorem 2.

Theorem 3. *Let G be a k-connected graph of order p. Let n be an integer with $0 \leqslant n \leqslant k$ and $p \equiv n \pmod 2$ and let α be a real number with $\frac{1}{2} \leqslant \alpha \leqslant 1$. If $|N_G(A)| > \alpha(p-2k+n-2)+k$ for every independent set A of order $\lfloor \alpha(k-n+2) \rfloor$, then G is n-factor-critical.*

Proof. Let $m = \lfloor \alpha(k-n+2) \rfloor$. Assume G is not n-factor-critical. Then $G-T$ has no perfect matching for some T with $|T| = n$. Then by Tutte's 1-factor theorem, $\mathrm{o}(G-T-S) > |S|$ for some $S \subset V(G)-T$. Since $\mathrm{o}(G-T-S)+|S| \equiv |V(G)-T| = p-n \equiv 0 \pmod 2$, $\mathrm{o}(G-T-S) \geqslant |S|+2$. Let $|S| = s$. Since $T \cup S$ separates G, $n+s \geqslant k$, or $s \geqslant k-n$.

Let $C_1, C_2, \ldots, C_t$ be the odd components of $G-T-S$. Then $t \geqslant s+2$. We may assume $|V(C_1)| \leqslant |V(C_2)| \leqslant \cdots \leqslant |V(C_t)|$. Since $t \geqslant s+2 \geqslant k-n+2 \geqslant m$, we can take $x_i \in V(C_i)$ $(1 \leqslant i \leqslant m)$. Let $A = \{x_1, x_2, \ldots, x_m\}$. Then A is an independent set. Let $U = \bigcup_{i=1}^t V(C_i)$ and $u = |U|$. Since $N_G(A) \subset (\bigcup_{i=1}^m V(C_i) - A) \cup S \cup T$, $|N_G(A)| \leqslant \sum_{i=1}^m |V(C_i)| - m + n + s$. By the assumption on the orders of $|V(C_i)|$'s, we have $\sum_{i=1}^m |V(C_i)| \leqslant (m/t)u \leqslant (m/(s+2))u$. Therefore,

$$|N_G(A)| \leqslant \frac{m}{s+2}u - m + s + n.$$

On the other hand, by the assumption

$$|N_G(A)| > \alpha(p-2k+n-2)+k \geqslant \alpha(u+s-2k+2n-2)+k.$$

Therefore, we have

$$\alpha(u+s-2k+2n-2)+k < \frac{m}{s+2}u - m + s + n,$$

from which we have

$$(\alpha(s+2)-m)u < (s+2)(s-m-k+n) - \alpha(s+2)(s-2k+2n-2).$$

Since $s \geqslant k-n$, $\alpha(s+2)-m \geqslant \alpha(k-n+2) - \lfloor \alpha(k-n+2) \rfloor \geqslant 0$. Furthermore, $u = \sum_{i=1}^t |V(C_i)| \geqslant t \geqslant s+2$. Therefore,

$$(\alpha(s+2)-m)(s+2) < (s+2)(s-m-k+n) - \alpha(s+2)(s-2k+2n-2)$$

or $2\alpha(s-k+n) < s-k+n$. This is a contradiction since $\alpha \geqslant \frac{1}{2}$ and $s-k+n \geqslant 0$. Therefore, the theorem follows. $\square$

The following corollary for matching extension is immediate from Theorem 3.

Corollary 4. *Let G be a k-connected graph of even order p. Let n be an integer with $0 \leqslant n \leqslant \frac{1}{2}k$ and α be a real number with $\frac{1}{2} \leqslant \alpha \leqslant 1$. If $|N_G(A)| > \alpha(p-2k+2n-2)+k$ for every independent set A of order $\lfloor \alpha(k-2n+2) \rfloor$, then G is n-extendable. In particular, if $|N_G(A)| \geqslant \frac{1}{2}p+n$ for every independent set A of order $\lfloor \frac{1}{2}(k-2n+2) \rfloor$, then G is n-extendable.*

Theorem 3 is sharp in the following sense. Let α be a rational number with $\frac{1}{2} \leqslant \alpha \leqslant 1$. Then we can write $\alpha = m/(k-n+2)$ for some nonnegative integers k, n and m with $n \leqslant k$. Let $G = K_k + (k-n+2)K_t$, where t is a positive odd integer. Then G is k-connected and $p = |V(G)| = k+(k-n+2)t \equiv n \pmod 2$. If A is an independent set of order $m = \alpha(k-n+2)$, then

$$|N_G(A)| \geqslant k+(t-1)m = k+\left(\frac{p-k}{k-n+2}-1\right)m = \alpha(p-2k+n-2)+k.$$

(The equality holds if $m \geqslant 2$.) However, G is not n-factor-critical.

We cannot drop the assumption $\alpha \geqslant \frac{1}{2}$ in Theorem 3. Let α be a real number such that $0 < \alpha < \frac{1}{2}$. Suppose that an integer $n \geqslant 0$ is given. Choose integers k and l such that $\lfloor \alpha(k-n+2) \rfloor \geqslant 2$ and $l > k$. Let $G = K_l + (l-n+2)K_1$. Since $l > k \geqslant n$, G is k-connected and G has order $p = 2l-n+2$ and hence $p \equiv n \pmod 2$. If A is an independent set of order $\lfloor \alpha(k-n+2) \rfloor \geqslant 2$, $|N_G(A)| = l$. On the other hand,

$$\alpha(p-2k+n-2)+k = 2\alpha(l-k)+k < l-k+k = l \leqslant |N_G(A)|,$$

since $\alpha < \frac{1}{2}$ and $l > k$. Therefore, G satisfies all the assumptions of Theorem 3 except for $\alpha \geqslant \frac{1}{2}$, while G is not n-factor-critical.

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